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Effect of Fluctuations on Electronic Properties above the Superconducting Transition*

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A calculation of the microscopic effects of the fluctuations of the superconducting order parameter in the classical region above T_c on the electron Green's function in dirty materials is presented. The calculation is formulated in the presence of a magnetic field. The results of this calculation in the zero-field limit are used to obtain the change in the thermal conductivity K and in the electron density of states $N(\omega)$. For a one-dimensional sample, K appears to diverge as $[(T-T_c)/T_c]^{-1/2}$ while for two and three dimensions, K depends weakly on temperature. The behavior of $N(\omega)$ is qualitatively similar to that of a gapless superconductor below its transition temperature, although the detailed frequency dependence is different. For thin films, the depression at the Fermi surface increases as $[(T-T_c)/T_c]^{-2}$. The agreement with experiment is satisfactory.

I. INTRODUCTION

RECENTLY, there has been great interest in the effects of thermodynamic fluctuations of the order parameter on various properties of a metal just above its superconducting transition temperature.[1–11] Thus far, most of the attention has focused on properties that directly involve the superfluid. For instance, in the presence of an electric field regions with finite superfluid density are freely accelerated, giving rise to anomalous behavior of the electrical conductivity.[1,2,5–10] It is natural to expect that fluctuations of the order parameter will also influence such properties of the single-particle excitations as the density of states,[3,11] the effective mass, and the lifetime. This, in turn, will also alter the electrical conductivity.[3,4] However, because the superfluid has a much stronger effect, the anomalous contribution from the normal fluid is not seen. In thermal conductivity, the situation is quite different. Since the superfluid does not contribute to the entropy flow, the normal electrons are dominant. A study of the thermal conductivity thus allows us to see the way the normal fluid is affected by fluctuations.

The influence of the fluctuations on the quasiparticles may also be observed in tunneling measurements.[11] We expect that just above the transition temperature, the quasiparticle spectrum will be altered by the presence of fluctuations and that this will be reflected in a variation of the tunneling density of states. The latter quantity is, in fact, a rather direct measure of the frequency and temperature dependence of the quasiparticle self-energy.

In this paper, we present a microscopic calculation of the electron Green's function in the presence of the fluctuations of the order parameter above the critical temperature T_c in the so-called classical region where mean-field theory is valid. We restrict ourselves to dirty materials where the fluctuation effects are usually most prominent. We use this calculation to obtain the change in thermal conductivity and electron density of states due to fluctuations. For completeness, we formulate the calculation in the presence of a magnetic field and we take the zero-field limit when it is convenient. Some of our results will therefore be applicable to thin films in a perpendicular field or to type-II superconductors.

In Sec. II, we calculate the electron self-energy with and without a magnetic field for various dimensionalities. We apply the result to the thermal conductivity in Sec. III. In Sec. IV, we calculate the electron density of states and compare our result with experiments.

II. ELECTRON SELF-ENERGY

The single-particle thermal Green's function will be written as

$$G=[i\omega-\xi_{\mathbf{k}}-\Sigma(\mathbf{k},\omega)]^{-1}, \tag{2.1}$$

where $\omega=\pi T(2n+1)$, $\xi_{\mathbf{k}}=k^2/2m-\epsilon_F$, and $\Sigma(\mathbf{k},\omega)$ is the self-energy for an excitation of wave number $\mathbf{k}$ and discrete Matsubara frequency ω. We use units in which

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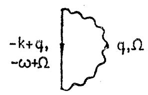

FIG. 1. Lowest-order diagram for the electron self-energy for wave vector **k** and discrete frequency ω. Solid line, electron Green's function; wavy line, fluctuation propagator.

Boltzmann's constant and $\hbar$ are unity. The lowest-order diagram for Σ is given in Fig. 1. The analytic expression for Σ corresponding to Fig. 1 is

$$\Sigma(\mathbf{k},\omega)=-\frac{T}{V}\sum_{\mathbf{q},\Omega} G_0(-\mathbf{k}+\mathbf{q},\ -\omega+\Omega)K(\mathbf{q},\Omega)\,, \quad (2.2)$$

where G_0 is the single-particle Green's function without Σ and $K(\mathbf{q},\Omega)$ is the propagator of the fluctuation field[12]:

$$\frac{1}{K(\mathbf{q},\Omega)}=N_0\left[\ln\frac{T}{T_{c0}}+\psi\left(\tfrac{1}{2}+\frac{|\Omega|+Dq^2}{4\pi T}\right)-\psi(\tfrac{1}{2})\right], \quad (2.3)$$

where $\psi(z)$ is the digamma function, T_{c0} is the transition temperature in zero magnetic field, N_0 is the density of states of unit spin at the Fermi surface of a free-electron gas in a volume V, and D is a diffusion constant.

In the presence of impurity scattering, the above expressions are modified as follows[13]: First, the discrete Matsubara frequency ω is replaced by $\tilde{\omega}=\omega+\Gamma_\omega$, where $\Gamma_\omega=(v_F/2l)\operatorname{sgn}\omega$, with $v_F=$ Fermi velocity and $l=$ mean free path. Second, the vertex of the interaction of a fluctuation line and two single-particle lines as shown in Fig. 2 is renormalized by the vertex correction Λ:

$$\begin{aligned}\Lambda_\omega(\mathbf{q},\Omega)&=(2\omega-\Omega+2\Gamma_\omega)/(2\omega-\Omega+D_\omega q^2)\,, && \text{if } \omega(\omega-\Omega)>0\\ &=1\,, && \text{if } \omega(\omega-\Omega)<0\,, \quad (2.4)\end{aligned}$$

where $D_\omega=D\operatorname{sgn}\omega$. Finally, in the dirty metal the diffusion constant has the value $D=\frac{1}{3}v_F l$.

If a uniform, constant magnetic field is present in the z direction, its effect is to modify[13] the fluctuation propagator, Eq. (2.3), and the vertex correction, Eq. (2.4), by the replacement $\mathbf{q}\rightarrow\mathbf{q}-2e\mathbf{A}/c$, where $\mathbf{A}=(0,Hx,0)$ is the vector potential.

With the above modifications, Eq. (2.2) becomes

$$\begin{aligned}\Sigma(\mathbf{k},\omega)=-\frac{T}{V}\sum_{\mathbf{q},\Omega} G_0(-\mathbf{k},\ -\omega+\Omega+\Gamma_{-\omega+\Omega})\\ \times\Lambda_\omega^2\left(\mathbf{q}-\frac{2e\mathbf{A}}{c},\ \Omega\right)K\left(\mathbf{q}-\frac{2e\mathbf{A}}{c},\ \Omega\right). \quad (2.5)\end{aligned}$$

A similar expression has been written down by Maki.[3] In Eq. (2.5), we have dropped the **q** dependence of G_0 since we expect the long-wavelength fluctuations to be most important. That is, we are neglecting $v_F q/\Gamma\approx l/\xi$, where ξ is the Ginzburg-Landau coherence length. In the expression for Σ, $\mathbf{q}-2e\mathbf{A}/c$ only appears in the form $L=D(\mathbf{q}-2e\mathbf{A}/c)^2$, which is the form of the Hamiltonian of a particle of charge $2e$ and mass $1/2D$ in a uniform magnetic field. It is therefore convenient to work in a representation of the eigenfunctions of this operator rather than in the plane-wave representation. The eigenvalues of L are $W_{nq}=2mD[\omega_c(n+\frac{1}{2})+q^2/2m]$, where $\omega_c=-2eH/mc$ and **q** is the wave vector of the motion along the z direction. When the sample is extensive in both the x and y directions, each of these eigenvalues is $\mathcal{A}m\omega_c/2\pi$-fold degenerate where $\mathcal{A}$ is the area of the sample perpendicular to H. We then find

$$\begin{aligned}\Sigma(\mathbf{k},\omega)=-\frac{T\mathcal{A}m\omega_c}{2\pi V}\sum_{n,q,\Omega} G_0(-\mathbf{k},\ -\omega+\Omega+\Gamma_{-\omega+\Omega})\\ \times\Lambda_\omega^2(n,q;\Omega)K_{nq}(\Omega)\,, \quad (2.6)\end{aligned}$$

where $\Lambda_\omega(n,q;\Omega)$ has the same form as Eq. (2.4) but with $D_\omega q^2$ replaced by $W_{nq}\operatorname{sgn}\omega$ and

$$\frac{1}{K_{nq}(\Omega)}=N_0\left[\ln\frac{T}{T_{c0}}+\psi\left(\tfrac{1}{2}+\frac{|\Omega|+W_{nq}}{4\pi T}\right)-\psi(\tfrac{1}{2})\right].$$

In subsequent applications, we shall be interested in the self-energy when its argument, the imaginary Matsubara frequency $i\omega$, is continued to the real axis. We wish to express the leading terms of Eq. (2.6) in a form suitable for this purpose. Some care must be exercised in achieving this. For example, the leading term is *not* found by simply setting $\Omega=0$. The relevant manipulations are described in Appendix A. From Eq. (A4), we find

$$\begin{aligned}\Sigma(\mathbf{k},\omega)=\frac{32T^2}{\pi VN_0}\frac{\tilde{\omega}^2}{i\tilde{\omega}+\xi}\frac{\mathcal{A}m\omega_c}{2\pi}\\ \times\sum_{q,n}(\Gamma_s+W_{nq})^{-1}(\Gamma_s+2W_{nq}+2\omega)^{-2}, \quad (2.7)\end{aligned}$$

where we have expanded the digamma function to first order in W_{nq}/T and we use the notation

$$\Gamma_s=(8T/\pi)\ln(T/T_{c0})$$

for the relaxation rate of the fluctuations of the order parameter. It is now possible to discuss various limiting cases. We shall always write

$$\Sigma(\mathbf{k},\omega)=(\tilde{\omega}^2/i\tilde{\omega}+\xi)S(\omega)\,. \quad (2.8)$$

$S(\omega)$ is then essentially the expansion parameter of our perturbation theory.

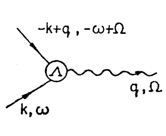

FIG. 2. Diagram for the pair-vertex function of wave vector q and discrete frequency Ω.

[12] C. Caroli and K. Maki, Phys. Rev. **159**, 306 (1967); **159**, 316 (1967).
[13] K. Maki, Phys. **1**, 21 (1964).

A. One Dimension, Zero Field

This case corresponds to samples in the shape of wires or whiskers whose cross section s is less than ξ^2, where $\xi(T)$ is the Ginzburg-Landau coherence length. It is convenient to take the long dimension parallel to z and to let ω_c become zero. The degeneracy $\mathcal{A}m\omega_c/2\pi$ is now unity and the only allowed n is zero. We find, from Eq. (2.8), that

$$S(\omega)=\frac{8T^2}{\pi N_0 s}\frac{1}{D^{1/2}}\left(\frac{2}{\Gamma_s}\right)^{5/2}\frac{1}{(\nu+1)^{3/2}}\frac{\sqrt{2}(\nu+1)^{1/2}+1}{[\sqrt{2}+(\nu+1)^{1/2}]^2},\tag{2.9}$$

where we have written $\omega=\frac{1}{2}\nu\Gamma_s$. The interesting behavior of this expression near T_c occurs because of the critical slowing down of Γ_s, which behaves as $\ln(T/T_c)\approx(T-T_c)/T_c=\epsilon$ for $T-T_c\ll T_c$.

B. Two Dimensions, Finite Field

In this case, we treat only the geometry of a thin film in a perpendicular field. Then, the only value of q which enters is zero and $W_{nq}=2mD\omega_c(n+\frac{1}{2})$, $\mathcal{A}/V=1/d$, where d is the film thickness. We find that

$$S(\omega)=\frac{T^2}{\pi^2 dN_0 D}\frac{1}{(mD\omega_c)^2}\frac{1}{2\bar{\omega}-\gamma}\bigg(\frac{2}{2\bar{\omega}-\gamma}\times[\psi(\tfrac{1}{2}+\tfrac{1}{2}\gamma+\bar{\omega})-\psi(\tfrac{1}{2}+\gamma)]-\psi^{(1)}(\tfrac{1}{2}+\tfrac{1}{2}\gamma+\bar{\omega})\bigg),\tag{2.10}$$

where $\gamma=\Gamma_s/2mD\omega_c$, $\bar{\omega}=\omega/2mD\omega_c$, and $\psi^{(1)}$ is the trigamma function.

In the perpendicular field H, the second-order transition occurs at a temperature $T<T_{c0}$ such that $\gamma=-\frac{1}{2}$. If we expand near this point for $\gamma+\frac{1}{2}=\eta$ small, we find[14]

$$S(\omega)=\frac{T^2}{\pi^2 dN_0 D}\frac{1}{(mD\omega_c)^2}\frac{2}{(2\bar{\omega}+\frac{1}{2})^2}\frac{1}{\eta}.\tag{2.11}$$

The critical behavior of this expression is made evident from the fact that

$$\eta=(H-H_{c2})/2H$$

for fixed T, $H>H_{c2}(T)$, and that

$$\eta=\frac{(2Tc/\pi DeH)(T-T_c)}{T_c}$$

for fixed H, $T>T_c$. The relation between H_{c2} and T (or T_c and H) is given by

$$\ln(T_{c0}/T)=\pi DeH_{c2}(T)/4cT.$$

[14] In the transition region, if $\rho=DeH/2\pi cT$ is not small, Eq. (2.2) is multiplied by $\frac{1}{2}\pi^2[\psi^{(1)}(1+\rho)]^{-1}$.

C. Two Dimensions, Zero Field

This case is simply found by passing to the limit $H=0$ in Eq. (2.10). The result is

$$S(\omega)=\frac{2T^2}{\pi^2 dN_0 D}\frac{1}{(\omega-\frac{1}{2}\Gamma_s)^2}\left(\ln\frac{\omega+\frac{1}{2}\Gamma_s}{\Gamma_s}-\frac{\omega-\frac{1}{2}\Gamma_s}{\omega+\frac{1}{2}\Gamma_s}\right).\tag{2.12}$$

D. Three Dimensions, Finite Field

Here, we consider the case of a type-II superconductor in which a second-order transition occurs, as in the case of a film, at $\gamma=-\frac{1}{2}$. This has already been treated by Maki.[3] Since in Eq. (2.7), we have treated W_{nq}/T as a small quantity, we are restricted to magnetic fields H such that $T_{c0}-T_c\ll T_{c0}$. On the other hand, it is possible, as Maki[3] has done, to simplify Eq. (A3) near the transition point by retaining only the term with $n=0$. If we do this, we obtain Maki's result after correcting an error in his Eq. (16) *et seq.*, where ϵ_0 should be replaced by $\frac{1}{2}\epsilon_0$.

E. Three Dimensions, Zero Field

In this case, it is most convenient to pass to the limit $H\to 0$ directly in Eq. (2.7) by treating n as a continuous variable. In this way, we find that

$$S(\omega)=\frac{T^2}{\pi^2 N_0 D^{3/2}}\frac{1}{(\omega-\frac{1}{2}\Gamma_s)^2}\frac{1}{(\omega+\frac{1}{2}\Gamma_s)^{1/2}}\times\{\tfrac{3}{2}\Gamma_s+\omega-2[\Gamma_s(\omega+\tfrac{1}{2}\Gamma_s)]^{1/2}\}.\tag{2.13}$$

Several of the results of this section will be used in our subsequent discussion.

We conclude this section with some remarks on the self-consistent calculation of the self-energy which proceeds from Eq. (2.2) with the modification that G_0 is replaced by G. This results in the following modification of Eq. (2.8):

$$\Sigma(\mathbf{k},\omega)=\frac{\omega^2}{i\bar{\omega}+\xi+\Sigma(-\mathbf{k},-\omega)}S(\omega),$$

from which we find that

$$2\Sigma(\mathbf{k},\omega)=-(i\bar{\omega}-\xi)\times\{[1+4\bar{\omega}^2 S(\omega)(\bar{\omega}^2+\xi^2)^{-1}]^{1/2}-1\}.\tag{2.14}$$

For small $S(\omega)$, Eq. (2.14) reduces to Eq. (2.8).

III. THERMAL CONDUCTIVITY

The thermal conductivity in the normal state is given by[15]

$$K=-\frac{1}{3T}\frac{d}{d\nu}\{\mathrm{Re}[P(\mathbf{q},\nu_m)|_{i\nu_m=\nu-i0^+}]\}|_{\mathbf{q}=0,\nu=0},\tag{3.1}$$

where $P(\mathbf{q},\nu_m)$ is the Fourier transform of the time-ordered heat-current–heat-current correlation function

[15] V. Ambegaokar and L. Tewordt, Phys. Rev. **134**, A805 (1964).

and $\nu_m = 2\pi T m$ runs over all integers. We will evaluate Eq. (3.1) in the Hartree-Fock approximation. Before we do this, let us point out that the diagram which is dominant in the electrical conductivity calculation, the one considered by Aslamazov and Larkin,[1] does not contribute to the thermal conductivity. The reason is that this diagram corresponds to the contribution of the superfluid flow to the current. Since the superfluid carries no entropy, it does not contribute to the thermal current.

The Hartree-Fock approximation corresponds to retaining only the scattering out terms in the Boltzmann equation. The relevant diagram is shown in Fig. 3. It can be shown that in this approximation, Eq. (3.1) takes the form

$$K = (24\pi m^2 T^2)^{-1} \sum_k \int d\omega\, k^2 \omega^2 \operatorname{sech}^2 \frac{\omega}{2T} A^2(\mathbf{k},\omega), \quad (3.2)$$

where $A(\mathbf{k},\omega)$ is the quasiparticle spectral function. From Eq. (3.2), we see that the most important contribution to K comes from the region $\omega \approx T_c$. The contribution from smaller frequencies is depressed by the ω^2 factor and also by the decreased density of states (cf. Sec. IV) which is reflected in $A(\mathbf{k},\omega)$. This means that, in the evaluation of Eq. (3.2), we can cut off the integral at small frequencies and the spectral function can be approximated by the results of the first-order perturbation theory that we have carried out in the previous section.

Using Eqs. (2.1) and (2.8), we can express the electron Green's function as

$$G(\mathbf{k},\omega) = -(i\tilde{\omega}_n + \xi)/\{\tilde{\omega}_n^2[1+S(\omega_n)] + \xi^2\}, \quad (3.3)$$

where $i\omega_n$ is the imaginary Matsubara frequency. The spectral function is related to the Green's function by

$$A(\mathbf{k},\omega) = 2\, \mathrm{Im} G(\mathbf{k},\, i\omega_n \to \omega + i0^+), \quad (3.4)$$

where in Eq. (3.4), ω is now a real frequency. Using Eqs. (3.3) and (3.4) in Eq. (3.2) and carrying out the integral over the kinetic energy, we obtain

$$K = \frac{v_F^2 N_0}{24T^2} \int d\omega\, \omega^2 \operatorname{sech}^2 \frac{\omega}{2T} \times \left[\frac{1}{2\omega\Gamma + \mathrm{Im}F} 2\, \mathrm{Re}\left(\frac{2\omega(\omega+i\Gamma)+F}{\epsilon_+}\right) + \mathrm{Im}\left(\frac{F}{\epsilon_+^3}\right)\right], \quad (3.5)$$

where $\Gamma = v_F/2l$ and we have written $F(\omega)$ for

$$(\omega + i\Gamma)^2 S(i\omega_n \to \omega + i0^+)$$

and where

$$\epsilon_+ = (\omega + i\Gamma)[1 + F(\omega)/(\omega+i\Gamma)^2]^{1/2}, \quad \mathrm{Im}\epsilon_+ > 0.$$

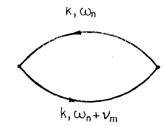

FIG. 3. Diagram for the Hartree-Fock approximation to the heat-current–heat-current correlation function of wave-vector zero and discrete frequency ν_m. Heavy lines, electron Green's function including self-energy due to fluctuations.

The thermal conductivity in two dimensions has only a weak logarithmic singularity as T approaches T_c and it has no anomalous behavior in three dimensions. Therefore, we treat one-dimensional geometry and we shall find that K appears to diverge as $[(T-T_c)/T_c]^{-1/2}$.

The size of the term in square brackets in Eq. (3.5) may be estimated by noticing that for our problem Γ_s/ω and ω/Γ are both small if we are sufficiently close to T_c. The ratio Γ_s/T_c is of order $\epsilon = (T-T_c)/T_c$, while T_c/Γ is of order 10^{-3}. It is convenient to introduce the quantity $\alpha(\omega)$, which is the approximation to $S(\omega)$ when $\omega/\Gamma_s \gg 1$. From Eq. (2.9), we find

$$\alpha = \frac{16T^2}{\pi N_0 \omega^2 s(\Gamma_s D)^{1/2}} = \frac{\epsilon_F}{T} \frac{8}{3ns\xi\epsilon}\left(\frac{T}{\omega}\right)^2,$$

where n is the electron density and ξ is the Ginzburg-Landau coherence length. Then $\alpha(\omega)$ is essentially the expansion parameter in the perturbation series. For $s \approx 10^{-10}$ cm^2, $\xi\epsilon^{1/2} \approx 10^{-5}$ cm, $\epsilon_F/T_c \approx 10^4$, and $n \approx 10^{22}$ cm^{-3}, we find that $\alpha \approx 10^{-3}(T_c/\omega)^2 \epsilon^{-1/2}$. For $\omega \approx T_c$, we see that α is small provided we are not too close to the transition. To first order in these parameters, we find that the expression in the curly brackets is $2(1+\frac{3}{2}\alpha)/\Gamma$. Since our use of Eq. (2.8) instead of Eq. (2.14) is correct only for small α, we should cut off the frequency integral in Eq. (3.4) so as to exclude the region where $\alpha > 1$. For the ratio $(K-K_0)/K_0$, we find a value 0.009 when $\epsilon = 10^{-2}$ and 0.07 when $\epsilon = 10^{-4}$, where K_0 is the normal-state conductivity in the absence of fluctuations. If we decrease the size of the sample, the ratio increases.

It is interesting to note that K increases in the presence of fluctuations. This suggests that the lifetime might increase. If we examine the imaginary part of the electron self-energy, we find that this is indeed the case. For $\omega \approx T_c$, we obtain $\mathrm{Im}\Sigma(\omega + i0^+) \approx -\Gamma(1-\alpha)$. This, together with the increase in the density of states ($\approx 1/2\alpha$ for $\omega \approx T_c$) accounts for the change in K. We remark that this affect on the lifetime occurs only in the dirty limit.

Thus far, we have considered just the Hartree-Fock approximation. To do a self-consistent calculation of the correlation function, we must take into account the change in the fluctuation propagator as well as the change in the electron Green's function.[16] Even if we assume the fluctuations to be unaffected by the thermal gradient, we still have to sum the ladder diagrams for the heat-current vertex function. However, because we have used Eq. (3.3) for the Green's function, the ladder series reduces to the term shown in Fig. 4. Instead of

[16] G. Baym and L. P. Kadanoff, Phys. Rev. 124, 287 (1961).

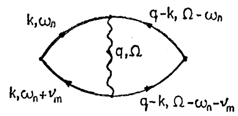

FIG. 4. Diagram for the scattering-in contribution to the heat-current–heat-current correlation function of wave-vector zero and discrete frequency ν_m. Heavy lines, electron Green's function including self-energy due to fluctuations; light lines, bare-electron Green's function.

calculating this diagram, we approximate all Green's functions by the bare ones (i.e., those without fluctuations but with impurity scattering). This corresponds to calculating the scattering-in terms by the first Born approximation. We hope that this will give the correct temperature dependence. In Appendix B, we show that in this approximation the scattering-in terms make no anomalous contribution to the thermal conductivity.

IV. DENSITY OF STATES

The electronic density of states in the transition region is easily obtained from the self-energy calculations of Sec. II. In the following, we shall only be interested in cases for which the expansion parameter $S(\omega)$ is small so that we may use Eq. (2.8) for the self-energy.

The density of states is found from the Green's function by the following expression:

$$N(\omega)=\frac{N_0}{\pi}\,\mathrm{Im}\int d\xi\, G(\mathbf{k},\, i\omega_n\to\omega+i0^+)\,. \qquad (4.1)$$

Using Eqs. (2.1) and (2.8), we find

$$\begin{aligned} N(\omega)&=N_0\,\mathrm{Re}[1+S(i\omega_n\to\omega+i0^+)]^{-1/2}\\ &\approx N_0[1-\tfrac{1}{2}\,\mathrm{Re}S(\omega)]\,, \end{aligned} \qquad (4.2)$$

where the second line is valid for small $S(\omega)$.

We apply our result to the case of two-dimensional films in zero magnetic field for which experiments[11] are available. From Eq. (2.12) we find, after continuing $i\omega_n\to\omega$, that

$$S(\omega)=-\frac{8T^2}{\pi^2N_0dD\Gamma_s{}^2}\left[\frac{1}{(\nu-i)^2}\left(\ln\frac{\nu+i}{2i}-\frac{\nu-i}{\nu+i}\right)\right], \qquad (4.3)$$

where $\nu=2\omega/\Gamma_s$. The term in square brackets is largest for $\nu<1$ and is of order unity so that the size of $S(\omega)$ is determined by the prefactor which may be written $-(3\pi^2/4m^2v_F{}^2ld\epsilon^2)$. Here, we have used the approximate expression $\Gamma_s=(8T_c/\pi)\epsilon$ which is valid for T close to T_c. For a film of thickness $d=550$ Å and mean free path $l=22$ Å (sample II of Ref. 11), we find that $S(\omega)$ is small as long as $\epsilon>10^{-2}$. Closer to the transition point it would be necessary to use the self-consistent result of Eq. (2.14) in the integral of Eq. (4.1). For temperatures such that Eqs. (4.2) and (4.3) are valid, we have the following behavior of $N(\omega)$: At the Fermi surface, there is a depression of $N(\omega)$ below N_0 by an amount which increases as ϵ^{-2} as $T\to T_c$. As ω increases, $N(\omega)$ increases (quadratically, for small ω) until it reaches N_0 at a frequency $\omega_0=1.14\Gamma_s$. For $\omega>\omega_0$, $N(\omega)$ continues to increase, there is a maximum at a frequency several times Γ_s and then $N(\omega)$ approaches N_0 as $\omega^{-2}\ln\omega$. Thus, the behavior is qualitatively similar to that of a gapless superconductor below its transition point,[17] although the detailed frequency dependence is different.

To compare our results with experiment, we focus our attention on the depression in $N(\omega)$ at zero frequency and on the frequency ω_0 at which $N(\omega)=N_0$. It is convenient for the former to consider $\delta N=[N(0)-N_0]/N_0$. From Eqs. (4.1) and (4.3), we find

$$\delta N=-3.0\times10^{-5}/\epsilon^2\,,$$

where we have used the parameters of sample II of Ref. 11. The ϵ^{-2} dependence agrees well with the data. The prefactor depends on the parameters of the granular aluminum film which are rather uncertain. The value ω_0, at which $N(\omega)=N_0$, is given by $1.14\Gamma_s=3.8\times10^{11}T_c\epsilon$ rad/sec and is otherwise sample-independent. The experimental results do not show enough regularity for comparison.

For a three-dimensional sample in zero field, we give only the result for δN. From Eqs. (2.13) and (4.2), we find that

$$\delta N=-T^2(3\sqrt{2}-4)/\pi^2N_0(D\Gamma_s)^{3/2}\,,$$

which varies as $\epsilon^{-3/2}$. The experimental results[11] appear to exhibit a slower variation.

For completeness, we give the result for $N(\omega)$ for a thin film in a perpendicular field. The bulk case has already been treated by Maki[3] (of Sec. II D). Near the transition point, we may use Eq. (2.11) for $S(\omega)$. From Eq. (4.2), we find that

$$\frac{N(\omega)-N_0}{N_0}=\frac{T^2}{\pi^2dN_0D}\,\frac{\omega^2-(eDH/c)^2}{[\omega^2+(eDH/c)^2]^2}\,\frac{1}{\eta}\,,$$

where η is defined below Eq. (2.11) and gives the critical temperature and field behavior. It is seen that, in this case, the density of states has the same form as that of a gapless superconductor (since only the $n=0$ fluctuation mode contributes for small η).

APPENDIX A: EVALUATION OF SELF-ENERGY

We wish to evaluate Eq. (2.6) for the electron self-energy in the presence of fluctuations. To achieve a form suitable for the analytic continuation of the positive imaginary frequency $i\omega$ to the real axis, we divide the sum on Ω into the ranges $\Omega>\omega$, $\omega>\Omega>0$, and $\Omega<0$. In the first range, the vertex correction Λ is unity since we are taking $\omega>0$. It is convenient to express the sum in

[17] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), p. 266.

each of the ranges over a positive summation variable. We find that

$$\Sigma(\mathbf{k},\omega)=-\frac{T\mathcal{A}m\omega_c}{2\pi V}\sum_{qn}\Big[\sum_{\omega'>0}G_0(\tilde{\omega}')K(\omega+\omega')$$
$$+\sum_{\omega>\omega'>0}G_0(-\tilde{\omega}')\Lambda_\omega^2(\omega-\omega')K(\omega-\omega')$$
$$+\sum_{\Omega>0}G_0(-\omega+\Omega+\Gamma_{-\omega+\Omega})\Lambda_\omega^2(-\Omega)K(\Omega)\Big],\quad \text{(A1)}$$

where we have not written the explicit dependence of the summands on $\mathbf{k}$, q, n.

In Eq. (A1), ω' is an odd discrete frequency while Ω is even. An examination of the summands reveals that the leading term of the first sum has a fluctuation propagator whose smallest frequency is of order T. On the other hand, the last sum contains a fluctuation propagator of zero frequency and its leading term will be larger by order $[\ln(T/T_{c0})]^{-1}$. Therefore, we neglect the sum and take $\Omega=0$ in the third. The second sum contains terms which can give a large contribution when $i\omega$ is continued to a real frequency less than T. To handle this, we rewrite the sum in such a way as to separate each factor into a partial fraction. The result is

$$-\frac{8T}{\pi N_0}\sum_{\omega>\omega'>0}\frac{(Z+2\Gamma-W)^2}{(L+Q)(L+Z)}\times\left[\frac{1}{L(L+Z)}+\frac{1}{Z^2}+\frac{1}{Z(L+Z)}\right],\quad \text{(A2)}$$

where

$$L=\frac{8T}{\pi}\left[\ln\frac{T}{T_{c0}}+\psi\left(\frac{1}{2}+\frac{\omega-\omega'+W}{4\pi T}\right)-\psi(\tfrac{1}{2})\right]$$

and $Q=i\tilde{\omega}'+\xi$, $Z=(\omega+\omega'+W)$. We have dropped all terms having more than one power of Q (order Γ) in the denominator as is consistent in the dirty limit.

We may neglect the first term in the square bracket of Eq. (A2) for the same reason we neglected the first sum of Eq. (A1). The other two terms are rewritten as the difference of two sums, $\infty>\omega'>0$ and $\infty>\omega'>\omega$. The largest contribution comes from the lower limit of the second sum which we add to the $\Omega=0$ part of the third sum of Eq. (A1). This yields the result

$$\Sigma(\mathbf{k},\omega)=\frac{T\mathcal{A}m\omega_c}{\pi^2VN_0}\frac{4\tilde{\omega}^2}{i\tilde{\omega}+\xi}\sum_{qn}\frac{1}{L(0)}\frac{1}{[L(0)+2\omega+W]^2},\quad \text{(A3)}$$

where we have neglected $L(0)$ in comparison with Γ.

APPENDIX B: SCATTERING-IN CONTRIBUTION TO K

Here, we calculate the contribution of the diagram of Fig. 4 to the thermal conductivity K. The correlation function $P(1,2)$ is given, in configuration space, by[15]

$$P(1,2)=\frac{1}{4m^2}\left(\frac{\partial}{\partial t_1}\nabla_{1'}+\frac{\partial}{\partial t_{1'}}\nabla_1\right)\left(\frac{\partial}{\partial t_2}\nabla_{2'}+\frac{\partial}{\partial t_{2'}}\nabla_2\right)\times\langle T\psi(1)\psi(2)\psi^\dagger(2')\psi^\dagger(1')\rangle|_{1'=1^+,2'=2^+},\quad \text{(B1)}$$

where ψ, $\psi^\dagger$ are the electron field operators and T is the time-ordering operator. The two lowest-order terms in the decomposition of the two-particle Green's function are

$$\langle T\psi(1)\psi(2)\psi^\dagger(2')\psi^\dagger(1')\rangle=G(12')G(21')+iG(1\bar{1})G(2\bar{1})K(\bar{1},\bar{2})G(\bar{2}1')G(\bar{2}2'),\quad \text{(B2)}$$

where $K(\bar{1},\bar{2})$ is the fluctuation propagator in configuration space. The first term on the right-hand side of Eq. (B2) is the Hartree-Fock approximation whose diagram is shown in Fig. 3. The second term is the first term in the ladder series. The relevant diagram is shown in Fig. 4, which depicts the quantity G_0G_0KGG, which is the correct self-consistent form for the scattering-in term when Eq. (3.3) is used for the Green's function. The contribution of Fig. 4 to the Fourier transform of the correlation function at zero wave number is

$$P_2(0,\nu_m)=-i(T/2m)^2\sum_{\mathbf{k},\mathbf{q},n}G_0(\mathbf{k},\omega_n)G_0(\mathbf{q}-\mathbf{k},\,-\omega_n)$$
$$\times G_0(\mathbf{k},\,\omega_n+\nu_m)G_0(\mathbf{q}-\mathbf{k},\,-\omega_n-\nu_m)$$
$$\times K(\mathbf{q},0)(2\omega_n+\nu_m)^2\mathbf{k}^2(\omega_n+\Gamma_n)/(\omega_n+\tfrac{1}{2}D_nq^2)$$
$$\times(\omega_n+\nu_m+\Gamma_{n+m})/(\omega_n+\nu_m+\tfrac{1}{2}D_{n+m}q^2),\quad \text{(B3)}$$

where $\Gamma_n(D_n)$ denotes $\Gamma\,\mathrm{sgn}\omega_n(D\,\mathrm{sgn}\omega_n)$. In writing Eq. (B3), we have followed Maki[18] and assumed that the dominant contribution is given by the zero-frequency-fluctuation propagator and, as discussed in the text, we have used bare Green's functions throughout. After performing the frequency sum and the sum on $\mathbf{k}$, analytically continuing $i\nu_m$ to $\nu+i0^+$, and neglecting v_Fq/Γ, we obtain

$$P_2(0,\nu)=i\frac{TN_0v_F^2}{24}\sum_{\mathbf{q}}\int dx\tanh\frac{x}{2T}\frac{(2x+\nu)^2}{(x+\nu-iDq^2)}\times[(\nu-2i\Gamma)(x+iDq^2)]^{-1}+[(2x+\nu-2i\Gamma)(x-iDq^2)]^{-1}]K(\mathbf{q},0)q^2.$$

We now neglect ν compared to Γ and find that

$$\frac{d}{d\nu}\mathrm{Re}P_2(0,\nu)=-\tfrac{1}{12}TN_0v_F^2D\textstyle\sum_{\mathbf{q}}K(\mathbf{q},0)\,\mathrm{Re}\displaystyle\int dx\times\tanh\frac{x}{2T}\left(\frac{4x^3Dq^2x^2}{(x^2+D^2q^4)^3[\Gamma^2+x^2]}\frac{(x+iDq^2)^3(x+i\Gamma)}{i\Gamma(x^2+D^2q^4)^3}\right).$$

We see that this expression does not diverge as $T\rightarrow T_c$.

[18] K. Maki, Progr. Theoret. Phys. (Kyoto) **40**, 193 (1968).